Student's Name: ______________________

Student Number:

Teacher's Name:

ABBOTSLEIGH

TEMPUS CELERIUS RADIO FUGIT

2021
HIGHER SCHOOL CERTIFICATE
Assessment Task 4

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen.
- **NESA approved** calculators may be used.
- **NESA approved** reference sheet is provided.
- All necessary working should be shown in every question.
- Make sure your HSC candidate Number is written at the top of every writing page.
- Please indicate the question and page number at the top of every page in the space provided (eg: Q 11, 1 of 4)
- Answer the Multiple-Choice questions on the answer sheet provided.
- Please write **'NOT ATTEMPTED'** for any questions not attempted.

Total marks - 100

- Attempt Sections I and II.

Section I — Pages 3 - 6

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section.

Section II — Pages 7 - 13

90 marks

- Attempt Questions 11 – 16.
- Allow about 2 hr and 45 minutes for this section.
- Marks are as indicated.

Outcomes to be assessed:

Mathematics Extension 2

HSC :
A student

MEX12-1 understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts

MEX12-2 chooses appropriate strategies to construct arguments and proofs in both practical and abstract settings

MEX12-3 uses vectors to model and solve problems in two and three dimensions

MEX12-4 uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems

MEX12-5 applies techniques of integration to structured and unstructured problems

MEX12-6 uses mechanics to model and solve practical problems

MEX12-7 applies various mathematical techniques and concepts to model and solve structured, unstructured and multi-step problems

MEX12-8 communicates and justifies abstract ideas and relationships using appropriate language, notation and logical argument

SECTION I

10 marks

Attempt Questions 1 – 10

Use the multiple-choice answer sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

(A) ○ (B) ● (C) ○ (D) ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

(A) ● (B) ~~●~~ (C) ○ (D) ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows.

correct

(A) ~~●~~ (B) ~~●~~ (C) ○ (D) ○

---

1. $\int \frac{3x - A}{1 - x^2}\, dx$ can be determined using the partial fractions $\frac{-1}{1+x} + \frac{2}{1-x}$.

The value of A is:

A. -3

B. -1

C. 1

D. 3

2. Mathematical induction is to be used to show $n(2n-1)(2n+1)$ is divisible by $3\ \forall\, n \in \mathbb{N}$.
The inductive step that requires proving is:

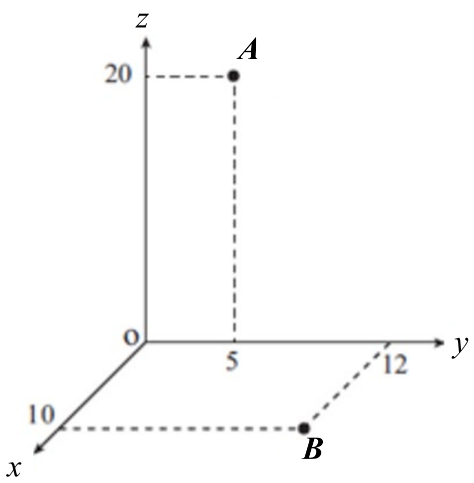
A. $(k+1)(2k)(2k+2)$ is divisible by 3.

B. $(k+1)(2k)(2k+3)$ is divisible by 3.

C. $(k+1)(2k+1)(2k+2)$ is divisible by 3.

D. $(k+1)(2k+1)(2k+3)$ is divisible by 3.

3. Consider points A and B as shown opposite.
The position vector representing the midpoint of AB is:



A. (5, 8.5, 10)

B. (5, 10, 8.5)

C. (10, 8.5, 5)

D. (10, 5, 8.5)

4. Given $z = 2 - 2i$ and $w = -3 + i$, calculate $z^2 - \overline{w}$.

A. $3 - 9i$

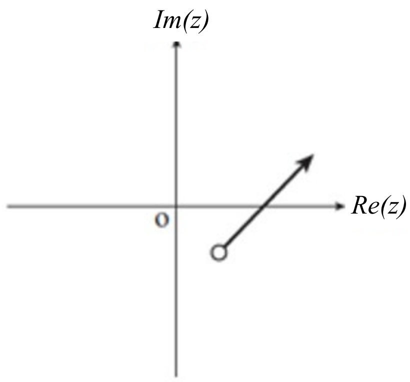
B. $3 - 7i$

C. $11 - 9i$

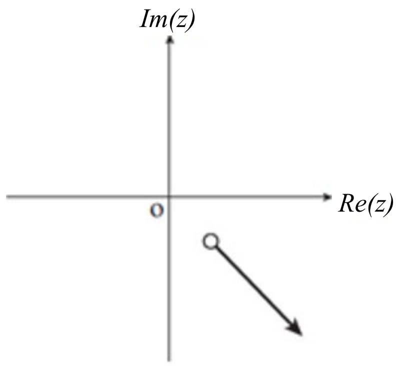
D. $11 - 7i$

5. The Argand diagram representing $\arg\left(z + i - 1\right) + \frac{\pi}{4} = 0$ is:

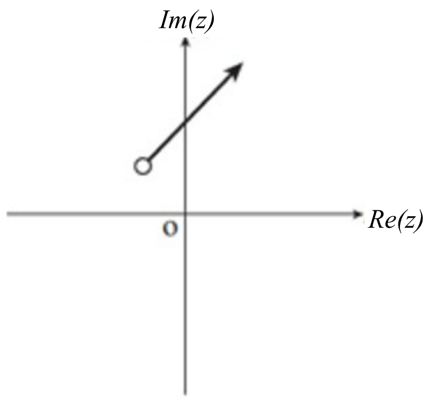
A.



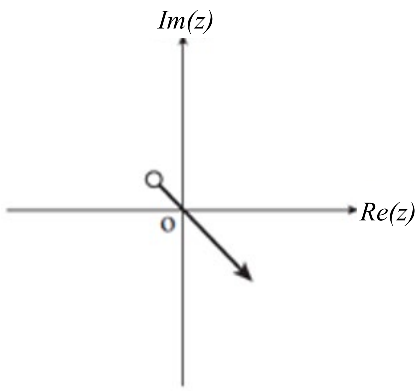
B.



C.

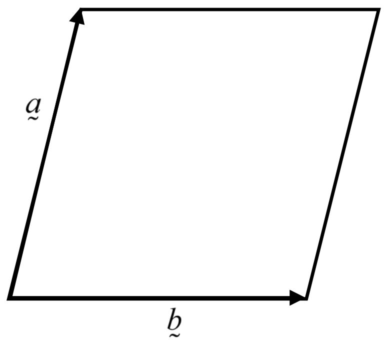


D.



6. The diagram opposite shows a rhombus.
Adjacent sides are represented by the two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

It follows that:



A. $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

B. $\underset{\sim}{a} = \underset{\sim}{b}$

C. $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = 0$

D. $|\underset{\sim}{a} + \underset{\sim}{b}| = |\underset{\sim}{a} - \underset{\sim}{b}|$

7. $1 + 2i$ is a root of the equation $z^3 - z^2 + qz + 5 = 0$. The value of q is:

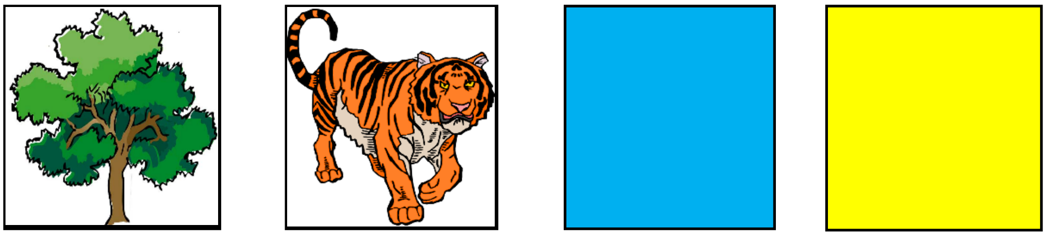
A. -3

B. -1

C. 3

D. 1

8. Four cards, each with a picture on one face and a plain colour on the other, are shown.



You are given the rule which states:

If a picture of a tree is on a card, then the colour yellow is on its other side.

The card showing the tree needs to be checked to prove this rule. Which of the following represents the best course to determine which other card (or cards) need to be turned over to check whether the rule holds?

A. Using proof by induction.

B. Using proof by contradiction.

C. Using proof by counterexample.

D. Using proof by contrapositive.

9. Given that $\frac{d}{dx}\left(x\cos^{-1}(x)\right)=\cos^{-1}(x)-\frac{x}{\sqrt{1-x^2}}$, it follows that $\int\cos^{-1}(x)\,dx$ is:

A. $x\cos^{-1}(x)+\int\frac{x}{\sqrt{1-x^2}}\,dx\ +C$

B. $\int x\cos^{-1}(x)\,dx\ +\ \frac{x}{\sqrt{1-x^2}}\ +C$

C. $x\cos^{-1}(x)-\frac{x}{\sqrt{1-x^2}}\ +C$

D. $\int x\cos^{-1}(x)\,dx\ \ +\int\frac{x}{\sqrt{1-x^2}}\,dx\ +C$

10. The exact value of $e^{\left(\frac{i\pi}{8}\right)}$ is:

A. $\sqrt[8]{i}$

B. $\sqrt[8]{-i}$

C. $\sqrt[4]{-i}$

D. $\sqrt[4]{i}$

End of Section I

SECTION II

Total Marks – 90
Attempt Questions 11 - 16
All questions are of equal value

Answer each question on the **SEPARATE** pages provided. Indicate at the top of the page the question and the page number (eg: Q11, 1 of 4).

In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = \dfrac{2 - 3i}{1 + i}$.

(i) Find $\overline{z}$ in the form $x + iy$. **2**

(ii) Evaluate $|z|$. **1**

(b) Find $\displaystyle\int \frac{1 + x}{4 + x^2}\,dx$. **2**

(c) Consider $w = -\sqrt{3} + i$.

(i) Express w in modulus-argument form. **2**

(ii) Hence or otherwise show that $w^7 + 64w = 0$. **2**

(d) Relative to a fixed origin O, the respective position vectors of three points A , B and C are:

$$\begin{pmatrix} 3 \\ 2 \\ 9 \end{pmatrix}, \begin{pmatrix} -5 \\ 11 \\ 6 \end{pmatrix} \text{ and } \begin{pmatrix} 4 \\ 0 \\ -8 \end{pmatrix}.$$

(i) Determine, in component form, the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. **2**

(ii) Hence find, to the nearest degree, $\angle BAC$. **2**

(iii) Calculate the area of ΔBAC. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Sketch the region in the complex plane where the inequalities

$1 \leq |z - i| \leq 2$ and $\text{Im}(z) \geq 0$ hold simultaneously.

Clearly mark in all *x and y* intercepts. **3**

(b) By completing the square find $\displaystyle\int \frac{1}{\sqrt{6 - x^2 - x}}\, dx$. **2**

(c) In an Argand diagram z is a point on the circle $|z| = 2$.

Given that $\arg z = \theta$ and $0 < \theta < \dfrac{\pi}{2}$

(i) Draw a diagram to represent this information. **1**

(ii) Find, in terms of θ, an expression for $\arg z^2$. **1**

(iii) Find, in terms of θ, giving brief reasons, expressions for:

(A) $\arg(z + 2)$. **1**

(B) $\arg(z - 2)$. **1**

(C) $\left|\dfrac{z - 2}{z + 2}\right|$. **1**

(d) Find $\displaystyle\int \sin^3 x \cos^3 x \; dx$. **2**

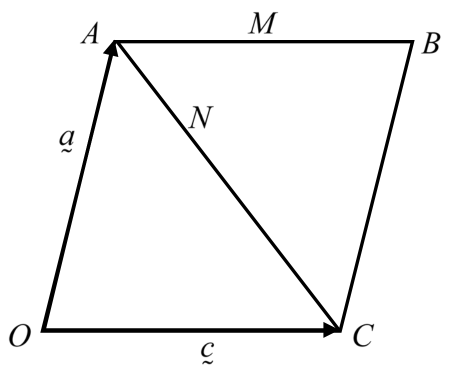
(e) Prove if $x, y \in \mathbb{Z}$, then $x^2 - 4y \neq 2$. **3**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\displaystyle\int_0^{\ln 2} x e^{-x} \, dx$. Give your answer in simplest form. **3**

(b) $OABC$ is a parallelogram and the point M is the midpoint of AB.
The point N lies on the diagonal AC so that $AN : NC = 1:2$.



NOT TO SCALE

Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

(i) Find simplified expressions, in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$, for each of the vectors $\overrightarrow{AC}$, $\overrightarrow{AN}$, $\overrightarrow{ON}$ and $\overrightarrow{NM}$. **4**

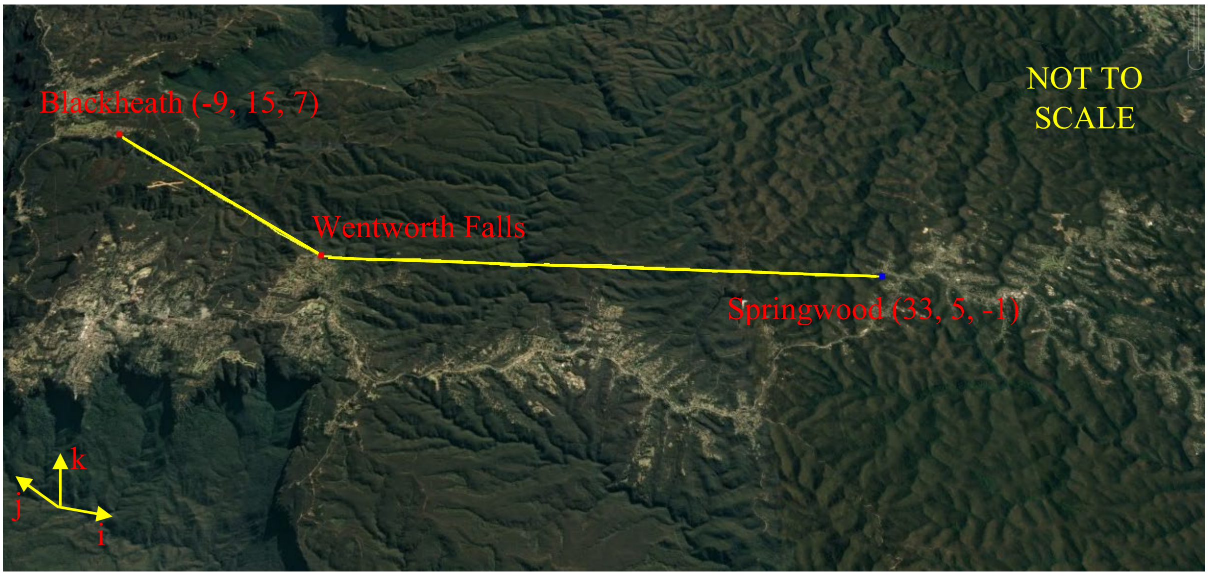
(ii) Deduce, showing your reasoning, that O, N and M are collinear. **1**

(c) By making the numerator rational, or otherwise, find $\displaystyle\int \sqrt{\frac{5-x}{5+x}} \, dx$ **3**

Question 13 continues on page 10

Question 13 (continued)

(d) Two tunnels are planned to be dug through Sydney's Blue mountains to improve traffic infrastructure.



Digging at one end of the tunnel is to begin at the point $(-9, 15, 7)$ at Blackheath and continue in the direction $7\mathbf{i} - 5\mathbf{j} - \mathbf{k}$. The digging at the other end of the tunnel will start at the coordinate $(33, 5, -1)$ near Springwood and continue in the direction $-14\mathbf{i} + 3\mathbf{k}$. Both sections are to be straight lines.

The coordinates are measured relative to a fixed origin O, where one unit is 500 metres.

(i) Show that the two sections of the tunnel will eventually meet at a point near Wentworth Falls and find the coordinates of this point. **2**

(ii) Find the total length of the two tunnels to the nearest kilometre. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Use a trigonometric substitution to find $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$ **3**

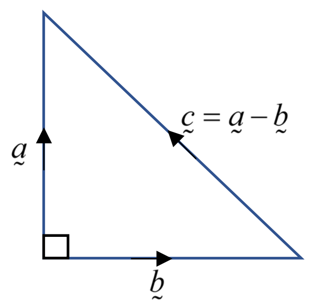
(b) Suppose that a_n $(n \ge 1)$ is a sequence defined by:

$$a_1 = 1, \quad a_2 = 3 \quad \text{and} \quad a_k = a_{k-1} + a_{k-2} \quad \forall\ k \ge 3.$$

Prove that $\forall\ n \ge 1$, we have $a_n \le \left(\dfrac{7}{4}\right)^n$. **3**

(c) A right-angled triangle is shown below, labelled with vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ where $\underset{\sim}{c} = \underset{\sim}{a} - \underset{\sim}{b}$.

Use these vectors and the dot product to prove Pythagoras' Theorem. **3**



NOT TO SCALE

(d) (i) Explain why the domain of the function, $f(x) = \sqrt{2-\sqrt{x}}$ is $0 \le x \le 4$. **1**

(ii) Show that $f(x)$ is a decreasing function and hence find its range. **2**

(iii) Using the substitution, $u = 2 - \sqrt{x}$ or otherwise, find the area bounded by the curve and the x and y axes. **3**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$ where n is an integer and $n \geq 3$.

(i) Show that $I_n + I_{n-2} = \dfrac{1}{n-1}$. **3**

(ii) Hence determine $\int_0^{\frac{\pi}{4}} \tan^4 x \, dx$. **1**

(b) (i) If $\dfrac{1}{x(\pi - 2x)} = \dfrac{A}{x} + \dfrac{B}{\pi - 2x}$ and $A = \dfrac{1}{\pi}$ find B in terms of π. **1**

(ii) Hence show that $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{x(\pi - 2x)} = \frac{2}{\pi} \ln 2$. **3**

(iii) By using the substitution $u = a + b - x$ show that:

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx.$$ **1**

(iv) Hence evaluate $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x \, dx}{x(\pi - 2x)}$ **3**

(c) Given $a^2 + b^2 \geq 2ab$ and $a + b + c = 1$, prove:

$$(1-a)(1-b)(1-c) \geq 8abc.$$ **3**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Prove for $k \in \mathbb{R}^+$: $\dfrac{2}{\sqrt{k+1}+\sqrt{k}} < \dfrac{1}{\sqrt{k}}$. **2**

(ii) Hence prove $16 < \displaystyle\sum_{k=1}^{80} \frac{1}{\sqrt{k}} < 17$. **4**

(b) (i) Solve $\tan 4\theta = 1$ for $0 \le \theta \le \pi$. **1**

(ii) Express $\tan 2\theta$ in terms of $\tan\theta$. **1**

(iii) Hence show $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$. **2**

(iv) Hence show $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ has roots

$$\tan\frac{\pi}{16},\ \tan\frac{5\pi}{16},\ \tan\frac{9\pi}{16} \text{ and } \tan\frac{13\pi}{16}.$$ **2**

(v) Hence evaluate $\tan\dfrac{\pi}{16}\tan\dfrac{5\pi}{16}\tan\dfrac{9\pi}{16}\tan\dfrac{13\pi}{16}$. **1**

(vi) Reduce the quartic equation in (iv) to a quadratic equation in terms of u where $u = x - \dfrac{1}{x}$ to then show:

$$\tan\frac{\pi}{16} - \cot\frac{\pi}{16} = -2 - 2\sqrt{2}.$$ **2**

End of Paper

① $-1(1-x)+2(1+x)$

$= -1+x+2+2x$

$= 3x+1$

$\therefore 3x - A = 3x+1$

$A = -1$ B

② For inductive step:

$(k+1)(2(k+1)-1)(2(k+1)+1)$

$= (k+1)(2k+1)(2k+3)$ D

③ $A(0,5,20)$ $B(10,12,0)$

$MP_{AB} = \left(\frac{0+10}{2}, \frac{5+12}{2}, \frac{20+0}{2}\right)$

$= (5, 8.5, 10)$ A

④ $z^2 - \bar{w} = (2-2i)^2 - (-3-i)$

$= -8i + 3 + i$

$= 3-7i$ B

⑤ $\arg(z-(1-i)) = -\pi/4$

From $(1,-1)$ with gradient -1

B

⑥ The diagonals in a rhombus intersect at right angles

Diagonals are $(\underset{\sim}{a}+\underset{\sim}{b})$ & $(\underset{\sim}{a}-\underset{\sim}{b})$

$\therefore (\underset{\sim}{a}+\underset{\sim}{b})\cdot(\underset{\sim}{a}-\underset{\sim}{b}) = 0$ C

⑦ The coefficients are real

Therefore, if $1+2i$ is a root, $1-2i$ is also a root (conjugate root theorem)

So, let the roots be $1+2i, 1-2i, \alpha$

By product of the roots:

$(1+2i)(1-2i)\alpha = -5$

$(1+4)\alpha = -5$

$\alpha = -1$

By sum of roots in pairs:

$(1+2i)(1-2i)+(1+2i)(-1)+(1-2i)(-1) = q$

$5 \quad -1-2i-1+2i = q$

$q = 3$ C

⑧ We need to turn the blue card over to show "if the colour on the other side is not yellow then the picture is not a tree

$\Rightarrow$ CONTRAPOSITIVE D

⑨ $\frac{d}{dx}(x\cos^{-1}x) = \cos^{-1}x - \frac{x}{\sqrt{1-x^2}}$

$\therefore \cos^{-1}x = \frac{d}{dx}(x\cos^{-1}x) + \frac{x}{\sqrt{1-x^2}}$

integrating w.r.t x

$\int \cos^{-1}x\,dx = \int \frac{d}{dx}(x\cos^{-1}x)\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx$

$\int \cos^{-1}x\,dx = x\cos^{-1}x + \int \frac{x}{\sqrt{1-x^2}}\,dx$ A

⑩ Consider $(e^{i\pi/8})^8 = e^{i\pi}$

$= -1.$

$\therefore \sqrt[8]{-1} = e^{i\pi/8}$

$\sqrt[8]{i^2} = e^{i\pi/8}$

$e^{i\pi/8} = \sqrt[4]{i}$ D.

QUESTION 11

a) $Z = \frac{2-3i}{1+i} \times \frac{1-i}{1-i}$ (1) Realise denom.

$= \frac{2-2i-3i+3i^2}{1^2+i^2}$

$= \frac{-1-5i}{2}$

$= -\frac{1}{2} - \frac{5}{2}i$

(i) $\bar{Z} = -\frac{1}{2} + \frac{5}{2}i$ (1)

(ii) $|Z| = \sqrt{(-\frac{1}{2})^2 + (-\frac{5}{2})^2}$

$= \frac{1}{2}\sqrt{1+25}$

$= \frac{\sqrt{26}}{2}$ (1)

b) $\int \frac{1+x}{4+x^2}\,dx = \int \frac{1}{4+x^2}\,dx + \frac{1}{2}\int \frac{2x}{4+x^2}\,dx$

$= \frac{1}{2}\tan^{-1}\frac{x}{2} + \frac{1}{2}\ln(4+x^2) + C$
(1) (1)

c) $w = -\sqrt{3} + i$ (2^{nd} Quadrant)

(i) $|w| = \sqrt{(-\sqrt{3})^2 + (1)^2}$

$= 2$ (1)

$\tan\alpha = \frac{1}{-\sqrt{3}}$

$\alpha = 5\pi/6$ (1)

$\therefore w = 2\,\text{cis}(\frac{5\pi}{6})$

(ii) $w^7 + 64w$

$= 2^7\,\text{cis}\,7(\frac{5\pi}{6}) + 64 \times 2\,\text{cis}\,\frac{5\pi}{6}$ (1)

$= 128\,\text{cis}\,\frac{35\pi}{6} + 128\,\text{cis}\,\frac{5\pi}{6}$

$= 128\,\text{cis}\,\frac{11\pi}{6} + 128\,\text{cis}\,\frac{5\pi}{6}$

$= 128\,\text{cis}\,-\frac{\pi}{6} + 128\,\text{cis}\,\frac{5\pi}{6}$

$= 128(+\cos\frac{\pi}{6} - i\sin\frac{\pi}{6} + -\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

$= 0$ (1) adjust angles.

d) (i) $\vec{AB} = \begin{pmatrix}-5\\11\\6\end{pmatrix} - \begin{pmatrix}3\\2\\9\end{pmatrix}$

$= \begin{pmatrix}-8\\9\\-3\end{pmatrix}$ (1)

$\vec{AC} = \begin{pmatrix}4\\0\\-8\end{pmatrix} - \begin{pmatrix}3\\2\\9\end{pmatrix}$

$= \begin{pmatrix}1\\-2\\-17\end{pmatrix}$ (1)

(ii) $\vec{AB} \cdot \vec{AC} = |\vec{AB}||\vec{AC}|\cos\angle BAC$ (1)

$\begin{pmatrix}-8\\9\\-3\end{pmatrix}\begin{pmatrix}1\\-2\\-17\end{pmatrix} = \sqrt{8^2+9^2+3^2}\,\sqrt{1^2+2^2+17^2}\,\cos\angle BAC$

$-8 - 18 + 51 = \sqrt{154}\,\sqrt{294}\,\cos\angle BAC$

$\cos\angle BAC = \frac{25}{\sqrt{154}\sqrt{294}}$

$\angle BAC = 83.2526\ldots$

$\angle BAC \doteqdot 83^\circ$ (1)

(iii) $A = \frac{1}{2}|\vec{AB}||\vec{AC}|\sin\angle BAC$

$= \frac{1}{2}\sqrt{154}\,\sqrt{294}\,\sin 83^\circ$ (1)

$= 105.7$ unit^2

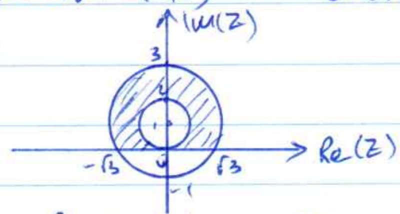
$\doteqdot 106$ unit^2 (1)

QUESTION 12

a) $1 \le |Z - i| \le 2$ $\qquad$ $\text{Im}(Z) \ge 0$

annulus centre $(0,1)$ $\qquad$ above real axis

(1) $\qquad$ (1)



for $x^2 + (y-1)^2 = 4$, when $y = 0$

$x^2 + 1 = 4$

$x^2 = 3$

$x = \pm\sqrt{3}$ (1) intercepts.

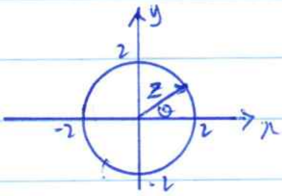
b) $\int \frac{1}{\sqrt{6-x^2-x}}\,dx = \int \frac{1}{\sqrt{6-(x^2+x+\frac{1}{4}-\frac{1}{4})}}\,dx$

$= \int \frac{dx}{\sqrt{\frac{25}{4} - (x+\frac{1}{2})^2}}$ (1)

$= \sin^{-1}\left(\frac{x+1/2}{5/2}\right) + C$

$= \sin^{-1}\left(\frac{2x+1}{5}\right) + C$ (1)

c) (i)

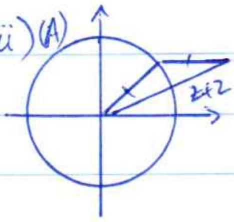


(1)

(ii) $\arg z^2 = 2\arg z$

$= 2\theta$ (1)

(iii)(A)

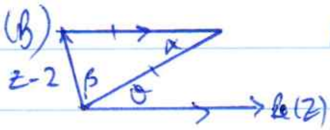


z+2 will be the diagonal of a rhombus with side length 2

$\therefore \arg(z+2) = \frac{1}{2}\arg(z)$

$= \frac{\theta}{2}$ (1)

(B)



$\alpha = \theta$ (alternate $\angle$ on $\parallel$)

$\therefore \beta = \frac{\pi - \alpha}{2}$ (isos $\triangle$ with side 2)

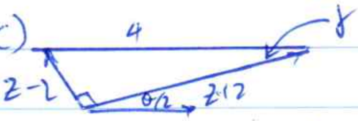
$= \frac{\pi}{2} - \frac{\theta}{2}$

$\therefore \arg(z-2) = \theta + \beta$

$= \theta + \frac{\pi}{2} - \frac{\theta}{2}$

$= \frac{\pi}{2} + \frac{\theta}{2}$ (1)

(C)



Note: the angle between z+2 & z-2 is $\text{Arg}(z-2) - \arg(z+2)$

$= \pi/2 + \frac{\theta}{2} - \theta/2$

$= \pi/2$

$\therefore$ we have a right angled $\triangle$ with $\gamma = \theta/2$ by alternate $\angle$

$\therefore \tan\gamma = \frac{|z-2|}{|z+2|}$

ie $\left|\frac{z+2}{z-2}\right| = \tan\frac{\theta}{2}$ (1)

d) $I = \int \sin^3 x \cos^3 x \, dx$

$= \int \sin^3 x \cos^2 x \cos x \, dx$

$= \int \sin^3 x (1 - \sin^2 x) \cos x \, dx$

$= \int (\sin^3 x - \sin^5 x) \cos x \, dx$ (1)

Let $u = \sin x$ $\therefore du = \cos x \, dx$.

$I = \int (u^3 - u^5) \, du$

$= \frac{u^4}{4} - \frac{u^6}{6} + C$

$= \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$ (1)

e) Prove, for $x, y \in \mathbb{Z}$, $x^2 - 4y \neq 2$

Proof by contradiction:

assume $\exists\, x, y \in \mathbb{Z}$ such that $x^2 - 4y = 2$ (1) METHOD

$\therefore x^2 = 4y + 2$

$x^2 = 2(2y+1)$

$\therefore x^2$ is even $\Rightarrow x$ is even. (1)

let $x = 2c$ for some $c \in \mathbb{Z}$

$\therefore (2c)^2 - 4y = 2$

$4c^2 - 4y = 2$

$c^2 - y = 1/2$.

but $c, y \in \mathbb{Z}$ $\therefore c^2, y \in \mathbb{Z}$

So $c^2 - y \neq 1/2$

CONTRADICTION. (1)

QUESTION 13

(1) SET-UP

a) $I = \int_0^{\ln 2} x e^{-x} \, dx$ let $u = x$ $v' = e^{-x}$

$u' = 1$ $v = -e^{-x}$

$\therefore I = \left[-xe^{-x}\right]_0^{\ln 2} - \int_0^{\ln 2} -e^{-x} \, dx$ (1)

$= -\ln 2\, e^{-\ln 2} - 0 + \left[-e^{-x}\right]_0^{\ln 2}$

$= -\ln 2\, e^{\ln 1/2} + -e^{-\ln 2} - -e^0$

$= -\frac{1}{2}\ln 2 + -\frac{1}{2} + 1$

$= \frac{1}{2}(1 - \ln 2)$ (1)

b) (i) $\vec{AC} = \vec{OC} - \vec{OA}$

$= \underset{\sim}{c} - \underset{\sim}{a}$ (1)

$\vec{AN} = \frac{1}{3}\vec{AC}$

$= \frac{1}{3}(\underset{\sim}{c} - \underset{\sim}{a})$ (1)

$\vec{ON} = \vec{OA} + \vec{AN}$

$= \underset{\sim}{a} + \frac{1}{3}(\underset{\sim}{c} - \underset{\sim}{a})$

$= \frac{1}{3}(2\underset{\sim}{a} + \underset{\sim}{c})$ (1)

$\vec{NM} = \vec{AM} - \vec{AN}$

$= \frac{1}{2}\underset{\sim}{c} - \frac{1}{3}(\underset{\sim}{c} - \underset{\sim}{a})$

$= \frac{1}{6}(3\underset{\sim}{c} - 2\underset{\sim}{c} + 2\underset{\sim}{a})$

$= \frac{1}{6}(2\underset{\sim}{a} + \underset{\sim}{c})$ (1)

(ii) The direction vector for $\vec{ON}$ & $\vec{NM}$ are the same: $(2\underset{\sim}{a} + \underset{\sim}{c})$
Since N is a common point, O, M & N are collinear (1)

c) $I = \int \sqrt{\frac{5-x}{5+x}}\, dx$

$= \int \frac{\sqrt{5-x}}{\sqrt{5+x}} \times \frac{\sqrt{5-x}}{\sqrt{5-x}}\, dx$

$= \int \frac{5-x}{\sqrt{25-x^2}}\, dx$ (1)

$= 5\int \frac{1}{\sqrt{25-x^2}}\, dx - -\frac{1}{2}\int \frac{-2x}{\sqrt{25-x^2}}\, dx$

$= 5\sin^{-1}\frac{x}{5} + \sqrt{25-x^2} + C$
(1) (1)

d) BW: $\begin{pmatrix} -9 \\ 15 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -5 \\ -1 \end{pmatrix}$

SW: $\begin{pmatrix} 33 \\ 5 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -14 \\ 0 \\ 3 \end{pmatrix}$

If the two tunnels meet:

$\underset{\sim}{i}$ $\quad -9 + 7\lambda = 33 - 14\mu$

$\underset{\sim}{j}$ $\quad 15 - 5\lambda = 5$

$\underset{\sim}{k}$ $\quad 7 - \lambda = -1 + 3\mu$. $\cancel{(1)}$

from $\underset{\sim}{j}$:

$15 - 5\lambda = 5$

$-5\lambda = -10$

$\lambda = 2$. (1) Find λ or μ

sub in $\underset{\sim}{k}$

$7 - 2 = -1 + 3\mu$.

$6 = 3\mu$

$\mu = 2$

confirm works for $\underset{\sim}{i}$

$-9 + 7(2) = 33 - 14(2)$

$-9 + 14 = 33 - 28$

$5 = 5$.

$\therefore$ Lines intersect at $\begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$ (1)

(ii) Total length:

$= \sqrt{(-9-5)^2 + (15-5)^2 + (7-5)^2}$
$+ \sqrt{(33-5)^2 + (5-5)^2 + (-1-5)^2}$

$= \sqrt{300} + \sqrt{820}$

$\doteqdot 45.956$

$\doteqdot 46$ (1)

Since 1 unit = 500 m = 0.5 km

Length = 45.956×0.5

$\doteqdot 23$ km. (1)

QUESTION 14

a) $I = \int \frac{dx}{x^2\sqrt{4-x^2}}$

Let $x = 2\sin\theta$ $\quad \therefore dx = 2\cos\theta\, d\theta$ (1)

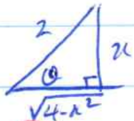
$I = \int \frac{2\cos\theta\, d\theta}{4\sin^2\theta\sqrt{4-4\sin^2\theta}}$

$= \int \frac{2\cos\theta\, d\theta}{4\sin^2\theta \cdot 2\cos\theta}$

$= \frac{1}{4}\int \operatorname{cosec}^2\theta\, d\theta$ (1)

So $I = -\frac{1}{4}\cot\theta + c$

$= -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + c$



$= -\frac{\sqrt{4-x^2}}{4x} + c$ 1

b) $a_1 = 1 \quad a_2 = 3 \qquad a_k = a_{k-1} + a_{k-2}$

Prove $a_n \le \left(\frac{7}{4}\right)^n$

Step 1: $a_1 = 1 \le \left(\frac{7}{4}\right)^1$

$a_2 = 3 \le \left(\frac{7}{2}\right)^2 = 3\frac{1}{16}$

$\therefore$ true for $n=1$ & $n=2$ 1

Step 2: Assume true for

$a_1, a_2, \dots, a_n$ for $k \le 2$

Hence $a_k \le \left(\frac{7}{4}\right)^k$

& $a_{k-1} \le \left(\frac{7}{4}\right)^{k-1}$

Step 3: Given the hypothesis in Step 2, prove true for $n = k+1$

i.e $a^{k+1} \le \left(\frac{7}{4}\right)^{k+1}$

$LHS = a_k + a_{k-1}$

$\le \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1}$ by hypothesis 1

$= \left(\frac{7}{4}\right)^{k-1}\left(\frac{7}{4}+1\right)$

$= \left(\frac{7}{4}\right)^{k-1}\left(\frac{11}{4}\right)$

$= \left(\frac{7}{4}\right)^{k-1}\left(\frac{44}{16}\right)$

$< \left(\frac{7}{4}\right)^{k-1}\left(\frac{49}{16}\right)$

$= \left(\frac{7}{4}\right)^{k-1}\left(\frac{7}{4}\right)^2$

$= \left(\frac{7}{4}\right)^{k+1}$ 1

Step 4: The result holds by the inductive process.

c) Note: $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$ as $\underset{\sim}{a} \perp \underset{\sim}{b}$

Given $\underset{\sim}{c} = (\underset{\sim}{a} - \underset{\sim}{b})$

$\underset{\sim}{c} \cdot \underset{\sim}{c} = (\underset{\sim}{a} - \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b})$ 1

$|\underset{\sim}{c}|^2 = \underset{\sim}{a} \cdot \underset{\sim}{a} - 2\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{b}$

$|\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$ as $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

1 as required. 1

d) (i) $f(x) = \sqrt{2-\sqrt{x}}$

for $\sqrt{x}$ $\quad x \ge 0$

for $\sqrt{2-\sqrt{x}}$ $\quad 2-\sqrt{x} \ge 0$

$\sqrt{x} \le 2$

$x \le 4$

$\therefore 0 \le x \le 4$ 1

(ii) $f'(x) = \frac{1}{2}(2-\sqrt{x})^{-1/2} \times -\frac{1}{2}x^{-1/2}$

$= \frac{-1}{4\sqrt{x(2-\sqrt{x})}} \qquad 0 < x < 4$

For $0 < x < 4$, $f'(x) < 0$

$\therefore f(x)$ is decreasing. 1

$f(0) = \sqrt{2} \qquad f(4) = 0$

$\therefore$ range $0 \le y \le \sqrt{2}$. 1

(iii) $A = \int_0^4 \sqrt{2-\sqrt{x}}\, dx$

Let $u = 2-\sqrt{x}$ $\qquad du = -\frac{1}{2}x^{-1/2}dx$

when $x=0$ $u=2$ $\qquad dx = -2(2-u)du$

$x=4$ $u=0$ $\qquad$ as $\sqrt{x} = 2-u$ 1

$\therefore A = \int_2^0 \sqrt{u}\,(-2(2-u))\, du$. 1

$= \int_0^2 \left(4u^{1/2} - 2u^{3/2}\right) du$

$= \left[\frac{8u^{3/2}}{3} - \frac{4u^{5/2}}{5}\right]_0^2$

$= \frac{16\sqrt{2}}{3} - \frac{16\sqrt{2}}{5}$

$= \frac{80\sqrt{2} - 48\sqrt{2}}{15}$

$= \frac{32\sqrt{2}}{15}$. 1

QUESTION 15

a) $I_n = \int_0^{\pi/4} \tan^n x \, dx \quad , \; n \geq 3$

(i) $I_n + I_{n-2} = \int_0^{\pi/4} \tan^n x \, dx - \int_0^{\pi/4} \tan^{n-2} x \, dx$

$= \int_0^{\pi/4} \tan^{n-2} x \left(\tan^2 x + 1 \right) dx$ 1

$= \int_0^{\pi/4} \tan^{n-2} x \left(\sec^2 x \right) dx$ 1

$= \left[\frac{1}{n-1} \tan^{n-1} x \right]_0^{\pi/4}$

$= \frac{1}{n-1} \left(\tan^{n-1}\left(\frac{\pi}{4}\right) - \tan^{n-1} 0 \right)$

$= \frac{1}{n-1} (1-0)$

$= \frac{1}{n-1}$ as required 1

(ii) Note $I_n = \frac{1}{n-1} - I_{n-2}$

Now $I_4 = \int_0^{\pi/4} \tan^4 x \, dx$

$I_4 = \frac{1}{4-1} - I_2$

$= \frac{1}{3} - \left(\frac{1}{2-1} - I_0 \right)$

$= \frac{1}{3} - (1) + \int_0^{\pi/4} dx$

$= \frac{1}{3} - 1 + \left[\frac{\pi}{4} - 0 \right]$

$= \frac{\pi}{4} - \frac{2}{3}.$ 1

b) (i) $\frac{1}{x(\pi - 2x)} = \frac{A}{x} + \frac{B}{\pi - 2x}$

If $A = \frac{1}{\pi}$ $\quad \frac{1}{\pi}(\pi - 2x) + Bx = 1$

Let $x = \pi/2$

$\therefore \; 0 + B\left(\frac{\pi}{2}\right) = 1$

$B = \cancel{\frac{\pi}{2}} \frac{2}{\pi}$ 1

(ii) $\int_{\pi/6}^{\pi/3} \frac{dx}{x(\pi - 2x)}$

$= \int_{\pi/6}^{\pi/3} \left(\frac{1/\pi}{x} + \frac{2/\pi}{\pi - 2x} \right) dx$ 1

$= \frac{1}{\pi} \int_{\pi/6}^{\pi/3} \left(\frac{1}{x} - \frac{-2}{\pi - 2x} \right) dx$

$= \frac{1}{\pi} \left[\ln x - \ln(\pi - 2x) \right]_{\pi/6}^{\pi/3}.$ 1

$= \frac{1}{\pi} \left[\ln \left(\frac{x}{\pi - 2x} \right) \right]_{\pi/6}^{\pi/3}.$

$= \frac{1}{\pi} \left[\ln \left(\frac{\pi/3}{\pi/3} \right) - \ln \left(\frac{\pi/6}{2\pi/3} \right) \right]$

$= \frac{1}{\pi} \left(0 - \ln\left(\frac{1}{4}\right) \right)$

$= \frac{1}{\pi} (2 \ln 2)$

$= \frac{2}{\pi} \ln 2$ 1

(iii) $I = \int_a^b f(a+b-x) \, dx$

Let $u = a + b - x$

$du = -dx.$

$x = a \Rightarrow u = b \qquad x = b \Rightarrow u = a.$

$\therefore \; I = \int_b^a f(u)(-du)$

$= \int_a^b f(u) \, du$

$= \int_a^b f(x) \, dx$ 1

(iv) consider $a = \pi/6, \; b = \pi/3.$

$I = \int_{\pi/6}^{\pi/3} \frac{\cos^2 x \, dx}{x(\pi - 2x)}$

$= \int_{\pi/6}^{\pi/3} \frac{\cos^2(\pi/6 + \pi/3 - x)}{(\pi/6 + \pi/3 - x)(\pi - 2(\pi/6 + \pi/3 - x))} dx.$ 1

$= \int_{\pi/6}^{\pi/3} \frac{\cos^2(\pi/2 - x)}{(\pi/2 - x)(\pi - 2(\pi/2 - x))} dx$

$= \int_{\pi/6}^{\pi/3} \frac{\sin^2 x}{(\pi/2 - x)(2x)} dx$

$= \int_{\pi/6}^{\pi/3} \frac{\sin^2 x}{(\pi - 2x) x} dx.$ 1

So $2I = \int_{\pi/6}^{\pi/3} \frac{\cos^2 x}{x(\pi - 2x)} dx + \int_{\pi/6}^{\pi/3} \frac{\sin^2 x}{x(\pi - 2x)} dx$

$= \int_{\pi/6}^{\pi/3} \frac{1}{x(\pi - 2x)} dx.$

$\therefore \; I = \frac{1}{2} \left(\frac{2}{\pi} \ln 2 \right)$ from (ii)

$= \frac{1}{\pi} \ln 2$ 1

6

c) Given $a^2+b^2 \geq 2ab$ & $a+b+c=1$

Prove $(1-a)(1-b)(1-c) \geq 8abc$

$LHS = (a+b+c-a)(a+b+c-b)(a+b+c-c)$ 1

$= (b+c)(a+c)(a+b)$

$= (ab+bc+ac+c^2)(a+b)$

$= a^2b + abc + a^2c + ac^2 + ab^2 + b^2c + abc + bc^2$

$= b(a^2+c^2) + c(a^2+b^2) + a(b^2+c^2) + 2abc$ 1

$\geq b(2ac) + c(2ab) + a(2bc) + 2abc$ applying $a^2+b^2 \geq 2ab$

$= 8abc$ as required. 1

QUESTION 16

a) (i) For $k \in R^+$ Prove $\dfrac{2}{\sqrt{k+1}+\sqrt{k}} < \dfrac{1}{\sqrt{k}}$

for $k \in R^+$ $\sqrt{k} < \sqrt{k+1}$ 1

$\sqrt{k}+\sqrt{k} < \sqrt{k+1} + \sqrt{k}$

$2\sqrt{k} < \sqrt{k+1} + \sqrt{k}$

$\sqrt{k} < \dfrac{\sqrt{k+1}+\sqrt{k}}{2}$

$\dfrac{1}{\sqrt{k}} > \dfrac{2}{\sqrt{k+1}+\sqrt{k}}$

or $\dfrac{2}{\sqrt{k+1}+\sqrt{k}} < \dfrac{1}{\sqrt{k}}$ # 1

(ii) Notice $\dfrac{2}{\sqrt{k+1}+\sqrt{k}} \times \dfrac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k+1}-\sqrt{k}}$

$= \dfrac{2(\sqrt{k+1}-\sqrt{k})}{k+1-k}$

$= 2(\sqrt{k+1}-\sqrt{k})$

$\therefore \dfrac{1}{\sqrt{k}} > 2(\sqrt{k+1}-\sqrt{k})$ 1

For $\sum_{k=1}^{80} \dfrac{1}{\sqrt{k}} > 16$:

$\sum_{k=1}^{80} \dfrac{1}{\sqrt{k}} = \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \dots + \dfrac{1}{\sqrt{79}} + \dfrac{1}{\sqrt{80}}$

$> 2(\sqrt{2}-\sqrt{1}) + 2(\sqrt{3}-\sqrt{2}) + \dots + 2(\sqrt{81}-\sqrt{80})$

$= -2\sqrt{1} + 2\sqrt{81}$

$= -2 + 18$

$= 16$

$\therefore \sum_{k=1}^{80} \dfrac{1}{\sqrt{k}} > 16$ 1

To show $\sum_{k=1}^{80} \dfrac{1}{\sqrt{k}} < 17$, derive another inequality: $\sqrt{k} > \sqrt{k-1}$

$2\sqrt{k} > \sqrt{k-1} + \sqrt{k}$

$\dfrac{2}{\sqrt{k-1}+\sqrt{k}} > \dfrac{1}{\sqrt{k}}$

$\dfrac{2}{\sqrt{k}+\sqrt{k-1}} > \dfrac{1}{\sqrt{k}}$ 1

$2(\sqrt{k}-\sqrt{k-1}) > \dfrac{1}{\sqrt{k}}$ (rationalising denominator.

Consider $\sum_{k=2}^{80} \dfrac{1}{\sqrt{k}} = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \dots + \dfrac{1}{\sqrt{80}}$

$< 2\left[(\sqrt{2}-\sqrt{1}) + (\sqrt{3}-\sqrt{2}) + \dots + \sqrt{80}-\sqrt{79}\right]$

$= 2\left[-\sqrt{1} + \sqrt{80}\right]$

$= 2\sqrt{80} - 2.$

$\therefore \sum_{k=1}^{80} \dfrac{1}{\sqrt{k}} = \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \dots + \dfrac{1}{\sqrt{80}}$

$< 1 + (2\sqrt{80}-2)$

< 17

$\therefore 16 < \sum_{k=1}^{80} \dfrac{1}{\sqrt{k}} < 17$ 1

b) (i) $\tan 4\theta = 1$ $0 \leq \theta \leq \pi$

Reference angle:

$4\theta = \pi/4$

$\therefore 4\theta = \pi/4, 5\pi/4, 9\pi/4, 13\pi/4,$

$\theta = \pi/16, 5\pi/16, 9\pi/16, 13\pi/16$ 1

(ii) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$ 1

(iii) $\tan 4\theta = \dfrac{2\tan 2\theta}{1-\tan^2 2\theta}$

$= \dfrac{2\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)}{1-\left[\frac{2\tan\theta}{1-\tan^2\theta}\right]^2}$ 1

$= \dfrac{\frac{4\tan\theta}{(1-\tan^2\theta)}}{\frac{(1-\tan^2\theta)^2-4\tan^2\theta}{(1-\tan^2\theta)^2}}$

$= \dfrac{4\tan\theta}{\cancel{1-\tan^2\theta}} \times \dfrac{(1-\tan^2\theta)^{\cancel{2}}}{1-2\tan^2\theta+\tan^4\theta-4\tan^2\theta}$

$= \dfrac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$ 1

(iv) let $x = \tan\theta$, $\tan 4\theta = 1$

$\therefore 1 = \dfrac{4x-4x^3}{1-6x^2+x^4}$ 1

$1-6x^2+x^4 = 4x-4x^3$

$x^4+4x^3-6x^2-4x+1=0$

so the solutions to this $eq^{\underline{n}}$ are the solutions to $\tan 4\theta = 1$ for $x = \tan\theta$ 1

$\therefore x = \tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, \tan\frac{9\pi}{16}, \tan\frac{13\pi}{16}$

(v) $\tan\frac{\pi}{16}\tan\frac{5\pi}{16}\tan\frac{9\pi}{16}\tan\frac{13\pi}{16} = -1$ by product of roots. 1

(vi) $x^4+4x^3-6x^2-4x+1=0$

$\div x^2,\ x^2 \neq 0$

$x^2+4x-6-\frac{4}{x}+\frac{1}{x^2}=0$

$x^2+\frac{1}{x^2}+4(x-\frac{1}{x})-6=0$

$(x^2-2+\frac{1}{x^2})+4(x-\frac{1}{x})-6+2=0$ 1

$(x-\frac{1}{x})^2+4(x-\frac{1}{x})-4=0$

Let $u = x-\frac{1}{x}$

$\therefore u^2+4u-4=0$

$u = \dfrac{-4\pm\sqrt{16+16}}{2}$

$= \dfrac{-4\pm4\sqrt{2}}{2}$

$= -2\pm2\sqrt{2}$

so $x-\frac{1}{x} = -2\pm2\sqrt{2}$

This will be negative when $|x|$ is least $\Rightarrow x = \tan\pi/16$ 1

$\therefore \tan\frac{\pi}{16} - \dfrac{1}{\tan\pi/16} = -2-2\sqrt{2}$

$\tan\frac{\pi}{16} - \cot\frac{\pi}{16} = -2-2\sqrt{2}.$